EXAMEN MATRICES Y SISTEMAS

Ejercicio 1

a) $A = (C_1, C_2, C_3) \in M_{3\times3}(\mathbb{R})$ $\quad \det(A) = \det(C_1, C_2, C_3) = 3$

$\det(B) = \det(C_1 + C_2, 2C_1 + 3C_3, C_2) = \det(C_1, 2C_1 + 3C_3, C_2) +$

$\det(C_2, \cancel{2C_1 + 3C_3, C_2})^{0} = \det(C_1, \cancel{2C_1, C_2})^{0} + \det(C_1, 3C_3, C_2) =$

2 columnas iguales $\qquad$ 2 columnas iguales

$= 3\det(C_1, C_3, C_2) = -3\det(C_1, C_2, C_3) = -3 \cdot 3 = -9$

$$\boxed{|B^{-1}| = \frac{1}{|B|} = \frac{1}{-9}}$$

b) A es invertible ($\exists A^{-1}$) $\longleftrightarrow$ $|A| \neq 0$

$|A| = \begin{vmatrix} a & 4+3a \\ 1 & a \end{vmatrix} = a^2 - 3a - 4$

$a^2 - 3a - 4 = 0 \begin{cases} a = 4 \\ a = -1 \end{cases}$ $\qquad$ $\boxed{\text{Luego } A \text{ invertible si } a \in \mathbb{R} - \{-1, 4\}}$

$a = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} \begin{matrix} \nearrow 4 \\ \searrow -1 \end{matrix}$

Si $a = 0$ $\quad A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$ $\quad |A| = -4 \neq 0 \rightarrow \exists A^{-1}$

$A^t = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$ $\quad \operatorname{Adj}(A^t) = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix} \rightarrow \boxed{A^{-1} = \frac{1}{-4}\begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}}$

Ejercicio 2:

a) $X \in M_{2\times 2}(\mathbb{R})$

$$B \cdot X + B = B^2 + I \rightarrow B \cdot X = B^2 + I - B$$

$$\underbrace{B^{-1} \cdot B}_{I} \cdot X = B^{-1}(B^2 + I - B) \rightarrow X = B^{-1}(B^2 + I - B)$$

$$B^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Cálculo B^{-1}:

$$|B| = -1 \neq 0 \rightarrow \exists B^{-1} \qquad B^t = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Adj(B^t) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\left(\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} = X}$$

b) $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ $\qquad BX = X \cdot B \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} x & y \\ x+z & y+t \end{pmatrix} = \begin{pmatrix} x+y & y \\ z+t & t \end{pmatrix} \rightarrow$$

$x = x + y \rightarrow y = 0$

$y = y$ $\qquad x = t$

$x + \cancel{z} = \cancel{z} + t$

$y + \cancel{t} = \cancel{t}$ $\qquad \forall z$

$$\boxed{X = \begin{pmatrix} x & 0 \\ z & x \end{pmatrix} \quad \forall x, z \in \mathbb{R}}$$

Ejercicio 3

a) $$\left.\begin{array}{l} x+ay+az=4 \\ ax+y-z=0 \\ 2x+2y-z=2 \end{array}\right\}$$

$$A^*=\begin{pmatrix} 1 & a & a & 4 \\ a & 1 & -1 & 0 \\ 2 & 2 & -1 & 2 \end{pmatrix} \in M_{4\times3}(\mathbb{R})$$

$$A=\begin{pmatrix} 1 & a & a \\ a & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$$

Rango de A ≤ 3

$$\begin{vmatrix} 1 & a & a \\ a & 1 & -1 \\ 2 & 2 & -1 \end{vmatrix} = -1+2a^2-2a-2a+2+a^2 = 3a^2-4a+1$$

$$3a^2-4a+1=0 \longrightarrow a=\frac{4\pm\sqrt{16-12}}{6}=\frac{4\pm2}{6} \begin{array}{l} a=1 \\ a=1/3 \end{array}$$

- Si $a\in\mathbb{R}-\{1, 1/3\}$ $Rg(A)=3$
- Si $a=1$ $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$ $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}=-2\neq0$ $Rg(A)=2$
- Si $a=1/3$ $A=\begin{pmatrix} 1 & 1/3 & 1/3 \\ 1/3 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$ $\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}=1\neq0$ $Rg(A)=2$

Rango de A^* ≤ 3

- Si $a\in\mathbb{R}-\{1, 1/3\}$ $(Rg(A^*)\geq3) \to Rg(A^*)=3$
- Si $a=1$ $A^*=\begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & -1 & 0 \\ 2 & 2 & -1 & 2 \end{pmatrix}$ $\begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 0 \\ 2 & -1 & 2 \end{vmatrix} = -2-4+8-2 = 0$

$$\begin{vmatrix} 1 & 1 & 4 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{vmatrix} \underset{C_1=C_2}{=} 0 \to Rg(A^*)<3 \to Rg(A^*)=2 \quad \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\neq0$$

• Si $a = 1/3$ $\quad A^* = \begin{pmatrix} 1 & 1/3 & 1/3 & 4 \\ 1/3 & 1 & -1 & 0 \\ 2 & 2 & -1 & 2 \end{pmatrix}$

$$\begin{vmatrix} 1/3 & 1/3 & 4 \\ 1 & -1 & 0 \\ 2 & -1 & 2 \end{vmatrix} = \cancel{2/3} - 4 + 8 - \cancel{2/3} = 4 \neq 0 \quad Rg(A^*) = 3$$

Tª Rouché- Fröbenius

	$a=1$	$a=1/3$	$a \in \mathbb{R} - \{1, 1/3\}$
$Rg(A)$	2	2	3
$Rg(A^*)$	2	3	3
Clasificación	SCI	SI	SCD
Nº soluciones	∞ sol	No sol	1 sol.

$n=3$

• Si $a = 1$ Sistema compatible indeterminado con 1 parámetro libre e ∞ soluciones

• Si $a = 1/3$ Sistema incompatible sin soluciones

• Si $a \in \mathbb{R} - \{1, 1/3\}$ Sistema compatible determinado con 1 solución

b) Si $a = 1 \rightarrow$ $\left.\begin{array}{l} x+y+z=4 \\ x+y-z=4 \end{array}\right\} \rightarrow$ $\begin{array}{r} y+z = 4-x \\ + \; y-z = 4-x \\ \hline 2y = 8-2x \end{array}$

$z = 4-x-y = \cancel{4}-x-\cancel{4}+x = 0$

$\boxed{\begin{array}{l} y = 4-x \\ z=0 \end{array} \quad \forall x \in \mathbb{R}}$

Ejercicio 4

$$\left.\begin{array}{l} x-2y+z=1 \\ x+y-az=1 \\ x+2y-2z=-2 \end{array}\right\} \qquad A^* = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & -a & 1 \\ 1 & 2 & -2 & -2 \end{pmatrix} \in M_{\cancel{4}\times 3}(\mathbb{R})$$

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -a \\ 1 & 2 & -2 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$$

Rango de A^* ≤ 3

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -a \\ 1 & 2 & -2 \end{vmatrix} = -\cancel{2} + \cancel{2} + 2a - 1 + 2a - 4 = 4a - 5$$

$4a-5=0 \qquad a = 5/4$

- Si $a \in \mathbb{R} - \{5/4\}$ $\quad Rg(A) = 3$
- Si $a = 5/4$ $\quad Rg(A) = 2$

$$\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$

Rango de A^* ≤ 3

- Si $a \in \mathbb{R} - \{5/4\}$ $\quad (Rg\, A \leq Rg\, A^*) \longrightarrow Rg(A^*) = 3$
- Si $a = 5/2$ $\quad A^* = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & -5/4 & 1 \\ 1 & 2 & -2 & -2 \end{pmatrix}$ $\quad Rg(A^*) = 3$

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = -2 + \cancel{2} + 2 - 1 - 2 + 4 = -1 \neq 0$$

Teorema Rouché - Fröbenius

	$a=5/4$	$a\in\mathbb{R}-5/4$
Rg(A)	2	3
Rg(A*)	3	3
Clasificación	S I	SCD
Nº soluciones	No solución	1 solución

$n=3$

- Si $a=5/4$ sistema incompatible, no soluciones
- Si $a\in\mathbb{R}-\{5/4\}$ Sistema compatible determinado, 1 solución

b) Si $a=1 \rightarrow$ SCD

$$\left.\begin{array}{l} x+2y+z=1 \\ x+y-z=1 \\ x+2y-2z=-2 \end{array}\right\}$$

Regla Cramer:

$$A^* = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 2 & -2 & -2 \end{pmatrix}$$

$$|A| = 4a-5 \neq -1$$

$a=1$

$$x = \frac{\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ -2 & 2 & -2 \end{vmatrix}}{-1} = \frac{\cancel{-2}+\cancel{2}-4+2+2-4}{-1} = 4$$

$$y = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & -2 \end{vmatrix}}{-1} = \frac{\cancel{-2}-2-1-1+2+\cancel{2}}{-1} = 6$$

$$z = \frac{\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -2 \end{vmatrix}}{-1} = \frac{\cancel{-2}+\cancel{2}-2-1-2-4}{-1} = 9$$

$$\boxed{\begin{array}{l} x=4 \\ y=6 \\ z=9 \end{array}}$$